Distributed control of networked Lur’e systems

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Chapter 6

Cooperative Robust Output Regulation of Heterogeneous Lur'e Networks

In this chapter, we study cooperative robust output regulation problems for a directed dynamical network of diffusively interconnected Lur'e systems that consist of a nominal linear dynamics with an unknown static nonlinearity around it. In the present chapter we assume that the nominal linear part of each agent is identical, but the nonlinearities are allowed to be different for distinct agents. In this sense the network is heterogeneous. As is common in the context of Lur'e systems, these unknown nonlinearities are assumed to be passive or sector bounded. The interconnection graph among these agents is assumed to contain a directed spanning tree. Similar to classical output regulation problems, there is an exosystem generating a common trajectory signal that these agents are required to track cooperatively. Our designed distributed dynamic state/output feedback protocol makes a copy of the common signal at each agent asymptotically, and thus the cooperative robust output regulation problem becomes a robust tracking problem that can be handled locally. It turns out that our cooperative protocols are naturally fully distributed. Sufficient conditions on state/output synchronizing protocols are given for both the passive unknown nonlinearity case and the sector bounded unknown nonlinearity case, respectively. Finally, two simulation examples illustrate our design.

6.1 Introduction

Indeed, multi-agent networks in the presence of uncertainties, non-uniform delays etc. are heterogeneous and their synchronization problems can usually be dealt with by means of robust control techniques. However, in practice, multi-agent networks are often intrinsically heterogeneous due to *non*-identical agent dynamics. In this case, output regulation theory and particularly the internal model principle have been explored to tackle such problem, see e.g. [66], where heterogeneous linear networks were studied. The main idea is that the models of the individual agents together with their local controllers must embed an internal model of the (virtual) exosystem that generates a common trajectory signal. In virtue of robust output regulation theory, output synchronization of heterogeneous uncertain linear networks can be also done, see e.g. [61]. Recently, in [12], the role of the internal model principle was discussed in the coordination of heterogeneous nonlinear networks.

Different from the aforementioned work, in this chapter we will study heterogeneous nonlinear networks, in which the dynamics of each agent is described by a Lur'e system. This Lur'e system is a nonlinear system consisting of a nominal linear dynamics with an unknown static nonlinearity around it. In our setting, these Lur'e systems are identical, in the sense that their linear parts are assumed to be identical and their unknown nonlinear parts are assumed to be passive or sector bounded within the same sector. However, the actual nonlinearities that occur are allowed to differ from agent to agent. In other words, the network is heterogeneous in the sense that the nonlinearities can be different for each agent. In Chapters 3-5, these unknown nonlinearities were assumed to be identical for each agent and satisfy the assumptions of *incremental* passivity and *incremental* sector boundedness, respectively. In this chapter, these nonlinearities are unknown but also, possibly, *non*-identical. Furthermore, only the conditions of passivity and sector boundedness are imposed. In addition, the interconnection topology can be directed but also time-varying. In this sense, our present results generalize those in Chapters 3-5. The case that both the nominal linear components as well as the unknown nonlinearities are non-identical can be considered in a similar, albeit technically more involved way, and is omitted from this chapter.

We stress that this chapter deals with the problem of agent dynamics with *functional uncertainties*. This setting is more challenging than that of only parametric uncertainties, which has been mostly considered in the literature. A convergence-based controller for robust output regulation of Lur'e systems was designed in [43]. The structure of the unknown nonlinearities is more general in the present chapter. In addition, in [43], the authors did not give an explicit description of the uncertainty involved, and robustness only holds with respect to some neighborhood of the nominal system. Cooperative output regulation of a class of nonidentical nonlinear systems in the form of a Lur'e system was discussed in [16]. There, besides the special structure of the nominal linear components, the nonlinear parts are assumed to be known precisely. Furthermore, in our opinion, it is not reasonable to assume that every agent has access to the output of the exosystem.

The rest of this chapter is organized as follows. In Section 6.2 the cooperative robust output regulation problem we deal with is formulated and some preliminaries are provided. Our solutions to the passive Lur'e-type nonlinearity case and the sector bounded Lur'e-type nonlinearity case are presented in Sections 6.3 and 6.4, respectively, along with some discussions in Section 6.5. A numerical simulation is given in Section 6.6. Some concluding remarks and possible future work close this chapter.

6.2 Problem formualtion

In this chapter we will consider a directed network of $N(\geqslant 2)$ identical Lur'e systems described by

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + Ed_i \\ y_i = Cx_i \\ z_i = Hx_i \\ d_i = -\phi_i(y_i) \end{cases}, \quad i = 1, 2, \cdots, N, \tag{6.1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $y_i(t) \in \mathbb{R}^p$ and $z_i(t) \in \mathbb{R}^q$ are the system state, the diffusive coupling input, the feedback loop input and the system output to be regulated of agent i, respectively. A, B, C, E and H are known constant matrices of compatible dimensions. Without loss of generality, we assume that the dimension m of the diffusive coupling inputs and the dimension q of the system outputs are strictly less than the state space dimension n. In this case the rows of matrix B are linearly dependent and thus $B^{\perp}$ exists. Similarly, $\left(H^T\right)^{\perp}$ exists as well. The equation $d_i = -\phi_i(y_i)$ represents a memoryless, nonlinear negative feedback loop, see Fig. 6.1. The function $\phi_i(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$ denotes an unknown static nonlinearity

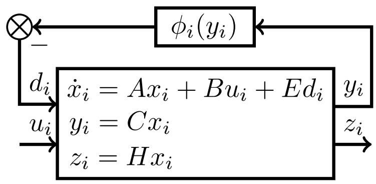

Figure 6.1: *Lur'e System*

around the nominal linear dynamics. In the context of Lur'e systems, the $\phi_i(\cdot)$'s are usually assumed to be passive, i.e.

$$y^T \phi_i(y) \geqslant 0, \ \forall \, y \in \mathbb{R}^p, \ i = 1, 2, \cdots, N,$$

or to be sector bounded, i.e.

$$(\phi_i(y) - S_1 y)^T (\phi_i(y) - S_2 y) \leqslant 0, \ \forall \, y \in \mathbb{R}^p, \ i = 1, 2, \cdots, N,$$

where the matrices S_1, $S_2 \in \mathbb{R}^{p \times p}$ satisfy $\mathbf{0} \leqslant S_1 < S_2$. For the SISO case, y and $\phi_i(\cdot)$ are scalars and hence $S_1 = \alpha$, $S_2 = \beta$ with $0 \leqslant \alpha < \beta$, see [73]. Obviously, sector boundedness is a more general property than passivity. The interconnection topology among the agents (6.1) is represented by a directed graph $\mathcal{G}$ that contains a directed spanning tree.

Remark 6.1. As stated in the introduction, in this chapter we consider networks of identical Lur'e systems, in the sense that the nominal linear parts and the *sets* in which the unknown static nonlinearities live are identical for each agent. More precisely, in Section 6.3 we will assume the identical set to be the set of all passive nonlinearities, and in Section 6.4 it will be the set of all sector bounded nonlinearities within a specified sector. In contrast with our previous work in Chapters 3-5 however, the actual nonlinearities that occur are allowed to differ for each of the agents. For this reason we use the notation $\phi_i(\cdot)$ to denote the nonlinearity in the dynamics of agent i.

Similar to classic output regulation problems [23], we assume that we have an exosystem given by

$$\dot{w} = Sw, \quad z = Rw, \tag{6.2}$$

where $w(t) \in \mathbb{R}^s$ and $z(t) \in \mathbb{R}^q$ are the state and the output, respectively. We assume that the matrix S has all its eigenvalues on the imaginary axis. The pair (S, R) is assumed to be detectable. For a given initial state of the exosystem, $z(t)$ is the reference signal that the agents (6.1) are required to track. In this sense the exosystem can be viewed as a *virtual leader*.

Remark 6.2. Different from the *self*-synchronization problems we studied before in e.g. [74], in the present chapter the synchronization manifold is specified in advance. We stress that in leader-following/pinning synchronization where a *free* agent in the network is selected to be the leader, the synchronization manifold defined by the trajectory of the leader is unknown if the dynamics of the leader is unknown, as for example in a Lur'e network.

In order to proceed, we will define an 'augmented graph' $\hat{\mathcal{G}}$ as follows. We introduce a new node, indexed by '0'. The dynamics associated with this new node is given by the exosystem (6.2). We assume that one of the root nodes in the original graph $\mathcal{G}$ has direct access to the output signal z of the exosystem. Without loss of generality, let '1' be the index of this root node. Then the augmented graph is given by $\hat{\mathcal{G}} = \{\{0\} \cup \mathcal{V}, (0,1) \cup \mathcal{E}\}$. The entry a_{10} of the associated adjacency matrix is defined to be equal to 1. Note that the node 0 is now the unique root node of the augmented graph $\hat{\mathcal{G}}$.

In this chapter we first assume that the agents (6.1) and the exosystem (6.2) can be interconnected by *dynamic state feedback* protocols of the form

$$\begin{cases} \dot{w}_1 = Sw_1 + T(z - Rw_1) \\ u_1 = Fx_1 + Kw_1 \end{cases}, \tag{6.3a}$$

$$\begin{cases} \dot{w}_i = Sw_i + \sum\limits_{j=1}^{N} a_{ij}(w_j - w_i) \\ u_i = Fx_i + Kw_i, \quad i = 2, \cdots, N, \end{cases} \tag{6.3b}$$

where $w_i(t) \in \mathbb{R}^s$ is the internal state of the dynamic protocol for agent i, T is a matrix such that $S - TR$ is Hurwitz, $\mathcal{A} = [a_{ij}]$ is the adjacency matrix associated with the graph $\mathcal{G}$, F and K are feedback and feedforward gain matrices to be determined, respectively.

Subsequently, we assume that the agents (6.1) and the exosystem (6.2) can be interconnected by *dynamic output feedback* protocols of the form

$$\begin{cases} \dot{w}_1 = Sw_1 + T(z - Rw_1) \\ \dot{w}_i = Sw_i + \sum\limits_{j=1}^{N} a_{ij}(w_j - w_i), \ i = 2, \cdots, N \end{cases}, \tag{6.4a}$$

$$\begin{cases} \dot{v}_i = A_c v_i + B_c(z_i - Rw_i) \\ u_i = C_c v_i + D_c z_i + Kw_i \end{cases}, \quad i = 1, 2, \cdots, N, \tag{6.4b}$$

where $w_i(t) \in \mathbb{R}^s$ and $v_i(t) \in \mathbb{R}^{n_c}$ are the internal states of the dynamic protocol for agent i, T is a matrix such that $S - TR$ is Hurwitz, $\mathcal{A} = [a_{ij}]$ is the adjacency matrix associated with the graph $\mathcal{G}$, A_c, B_c, C_c, D_c and K are gain matrices to be determined, respectively. For the dynamic output feedback protocol design, H is assumed to have full row rank.

Remark 6.3. We want to stress that the dynamic state feedback protocol (6.3a)-(6.3b) and the dynamic output feeedback protocol (6.4a)-(6.4b) share the same estimator for the state of the exosystem (6.2). In fact, the interconnections among the agents happen in the distributed estimator network. Such framework is commonly used to deal with heterogeneous networks. This paradigm can be found in Fig. 2 in [12].

In the above settings, we study the cooperative robust output regulation problems:

Problem: The network of agents (6.1) with the protocol (6.3a)-(6.3b) and respectively, (6.4a)-(6.4b) is robustly output regulated if $z_i(t) - z(t) \to \mathbf{0}$ as $t \to \infty$ for all initial conditions on the exosystem and the agents, and all nonlinearities $\phi_i(\cdot)$, $\forall\, i = 1, 2, \cdots, N$, satisfying the passivity or sector boundedness condition.

Before moving on, a basic preliminary result will be given below.

Lemma 6.4. In (6.3a)-(6.3b) and similarly (6.4a), $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, $\forall\, i = 1, 2, \cdots, N$, for all initial conditions on the exosystem and the protocol.

Proof. Let $\tilde{w}_i = w_i - w$, $\forall\, i = 1, 2, \cdots, N$. We have

$$\dot{\tilde{w}}_1 = (S - TR)\tilde{w}_1,$$

where $S - TR$ is Hurwitz. Obviously, $\tilde{w}_1(t)$ goes to **0** exponentially. On the other hand, we have

$$\dot{\tilde{w}}_i = S\tilde{w}_i + \sum_{j=1}^{N} a_{ij} \left(\tilde{w}_j - \tilde{w}_i\right), \quad i = 2, \cdots, N,$$

i.e.

$$\dot{\tilde{w}} = \left(I_{N-1} \otimes S - \tilde{L} \otimes I_s\right) \tilde{w} - l_{21} \otimes \tilde{w}_1,$$

where $\tilde{w} = \left[\tilde{w}_2^T, \cdots, \tilde{w}_N^T\right]^T$, $\mathcal{L} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & \tilde{L} \end{bmatrix}$. Here, $\mathcal{L}$ is the Laplacian matrix associated with the graph $\mathcal{G}$. Since agent 1 does not use the relative information with respect to other agents, the Laplacian matrix describing the interconnection relations *in the protocol* is given by $\tilde{\mathcal{L}} = \begin{bmatrix} 0 & \mathbf{0} \\ l_{21} & \tilde{L} \end{bmatrix}$. It is easily seen that $\tilde{\mathcal{L}}$ has a unique zero eigenvalue and its other eigenvalues have strictly positive real parts. Therefore, $-\tilde{L}$ is Hurwitz. Let $v(t) = \left(I_{N-1} \otimes e^{-St}\right) \tilde{w}(t)$. We get

$$\begin{aligned}
\dot{v} = &- \left(I_{N-1} \otimes Se^{-St}\right) \tilde{w} + \left(I_{N-1} \otimes e^{-St}\right) \\
&\left[\left(I_{N-1} \otimes S - \tilde{L} \otimes I_s\right) \tilde{w} - l_{21} \otimes \tilde{w}_1\right] \\
= &- \left(\tilde{L} \otimes e^{-St}\right) \tilde{w} - l_{21} \otimes \left(e^{-St}\tilde{w}_1\right) \\
= &- \left(\tilde{L} \otimes I_s\right) v - l_{21} \otimes \left(e^{-St}\tilde{w}_1\right).
\end{aligned}$$

Since S has all its eigenvalues on the imaginary axis and $\tilde{w}_1$ vanishes exponentially, $e^{-St}\tilde{w}_1$ goes to zero exponentially. In addition, $-\tilde{L} \otimes I_s$ is Hurwitz. Thus $\tilde{w}$ as well as v tends to zero exponentially. This completes the proof. □

Remark 6.5. Similar to [60], we can also consider time-varying topologies. In fact, any form of topology would be possible here provided that we can achieve $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ globally, $\forall\, i = 1, 2, \cdots, N$.

In the next two sections we will discuss cooperative robust output regulation of the network of agents (6.1) with the protocols (6.3a)-(6.3b) and (6.4a)-(6.4b), respectively. In Section 6.3, the uncertainties in the negative feedback loops are modeled by assuming the unknown nonlinear functions $\phi_i(\cdot)$'s to be passive. In Section 6.4, we model the uncertainties by assuming that the functions $\phi_i(\cdot)$'s satisfy a sector boundedness condition.

6.3 Passive nonlinearities

In this section, the unknown feedback nonlinearities $\phi_i(\cdot)$'s are assumed to be passive. Here we want to stress that in [74], the feedback nonlinearities were assumed to be identical and *incrementally* passive. First we will now study the dynamic state feedback protocol (6.3a)-(6.3b). Later on, the dynamic output feedback protocol (6.4a)-(6.4b) will be discussed.

6.3.1 Dynamic state feedback

Theorem 6.6. Assume that $\phi_i(\cdot)$'s, $\forall\, i = 1, 2, \cdots, N$, are passive. Let Π and Γ be a solution pair to the regulator equations

$$\begin{cases} \Pi S = A\Pi + B\Gamma \\ \mathbf{0} = C\Pi \\ \mathbf{0} = H\Pi - R \end{cases}. \tag{6.5}$$

If there exists a positive definite matrix P and a matrix F such that

$$\begin{cases} P(A + BF) + (A + BF)^T P < \mathbf{0} \\ PE = C^T \end{cases}, \tag{6.6}$$

then the network of agents (6.1) with protocol (6.3a)-(6.3b), where $K = \Gamma - F\Pi$, is robustly output regulated.

Proof. Let $\tilde{x}_i = x_i - \Pi w_i$, $i = 1, 2, \cdots, N$, where Π together with Γ satisfies (6.5). We get

$$\dot{\tilde{x}}_1 = (A + BF)\tilde{x}_1 - E\phi_1(C\tilde{x}_1) - \Pi T(z - Rw_1), \tag{6.7a}$$

$$\dot{\tilde{x}}_i = (A + BF)\tilde{x}_i - E\phi_i(C\tilde{x}_i) - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i),\ i = 2, \cdots, N. \tag{6.7b}$$

Denote $\Sigma_1 := \Pi T(z - Rw_1)$ and $\Sigma_i := \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i)$, $i = 2, \cdots, N$. By Lemma 6.4, $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, $i = 1, 2, \cdots, N$.

Consider a Lyapunov function candidate $V_1(\tilde{x}_i) = \tilde{x}_i^T P \tilde{x}_i$, $i = 1, 2, \cdots, N$, where $P > \mathbf{0}$ together with F satisfies (6.6). Obviously, $V_1(\tilde{x}_i)$ is positive definite and radially unbounded. Then the time derivative of $V_1(\tilde{x}_i)$ along the trajectories of (6.7a)-(6.7b) is given by

$$\dot{V}_1(\tilde{x}_i) = 2\tilde{x}_i^T P[(A + BF)\tilde{x}_i - E\phi_i(C\tilde{x}_i) - \Sigma_i]$$

$$
\begin{aligned}
&= \tilde{x}_i^T \left[P(A+BF) + (A+BF)^T P\right] \tilde{x}_i \\
&\quad - 2(C\tilde{x}_i)^T \phi_i(C\tilde{x}_i) - 2\tilde{x}_i^T P\Sigma_i \\
&\leqslant \tilde{x}_i^T \left[P(A+BF) + (A+BF)^T P\right] \tilde{x}_i - 2\tilde{x}_i^T P\Sigma_i,
\end{aligned}
$$

where we have used the property of passivity of each $\phi_i(\cdot)$. We note that there always exists a positive real number α such that

$$
\begin{aligned}
\dot{V}_1(\tilde{x}_i) \leqslant &- \alpha\tilde{x}_i^T P\tilde{x}_i - 2\tilde{x}_i^T P\Sigma_i \\
\leqslant &- \alpha\lambda_{\min}(P)||\tilde{x}_i||^2 + 2||\tilde{x}_i||||P\Sigma_i|| \\
\leqslant &- \alpha\lambda_{\min}(P)||\tilde{x}_i||^2 + \beta||\tilde{x}_i||^2 + \frac{1}{\beta}||P\Sigma_i||^2 \\
= &- (\alpha\lambda_{\min}(P) - \beta)||\tilde{x}_i||^2 + \frac{1}{\beta}||P\Sigma_i||^2,
\end{aligned}
$$

where $0 < \beta < \alpha\lambda_{\min}(P)$. Obviously, the systems (6.7a)-(6.7b) are globally input-to-state stable with Σ_i, $i = 1, 2, \cdots, N$, as the inputs, respectively. Since $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, by taking Σ_i as the output of a globally exponentially stable linear system with zero input, it is easily seen that $\tilde{x}_i(t)$ goes to zero as $t \to \infty$. Together with $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, the proof is complete. □

Remark 6.7. From the above analysis it is clear how the case of *non*-identical nominal linear parts can also be tackled. In this case, a set of N *non*-identical regulator equations will be required to have a common solution pair, see e.g. (6) in [66]. In addition, similar to classic output regulation problems, disturbance rejection at each agent can be considered, which just introduces more regulator equations [12]. The solution of this problem is also left to the reader.

Note that Theorem 6.6 does not tell us how to compute a suitable F for a given Lur'e network, and consequently a suitable K is unknown either. Referring to our previous work, particularly Lemma 3.3 in Chapter 3, the following result complements Theorem 6.6 by giving a suitable F and subsequently a suitable K.

Lemma 6.8. There exists a positive definite matrix P and a matrix F such that (6.6) holds if and only if there exists a positive definite matrix Q such that

$$
\begin{cases} B^{\perp}\left(QA^T + AQ\right)\left(B^{\perp}\right)^T < \mathbf{0} \\ E = QC^T \end{cases} . \tag{6.8}
$$

In this case, a suitable P is given by $P = Q^{-1}$, and a suitable F is given by $F = \mu B^T Q^{-1}$, where μ is any real number satisfying $QA^T + AQ + 2\mu BB^T < \mathbf{0}$.

Proof. For a proof we refer to Lemma 3.3 in Chapter 3. □

Thus, the computation of a cooperative robust output regulation protocol can be performed as follows:

- Compute a solution pair (Π, Γ) to (6.5);
- Compute a $Q > \mathbf{0}$ such that (6.8) holds;
- Compute a μ such that $QA^T + AQ + 2\mu BB^T < \mathbf{0}$;
- Compute $F := \mu B^T Q^{-1}$;
- Compute $K := \Gamma - F\Pi$.

Remark 6.9. Note that in the above computation, the knowledge of the entire interconnection topology, which is a kind of global information, is not required. This is in contrast with our work in [74] where we had to employ an adaptive protocol to remove such requirement. In this sense, our designed protocol in the present chapter is *naturally* fully distributed thanks to the result of Lemma 6.4.

6.3.2 Dynamic output feedback

In this subsection, we will discuss the design of a dynamic output feedback protocol in the form of (6.4a)-(6.4b).

Theorem 6.10. Assume that $\phi_i(\cdot)$'s, $\forall\, i = 1, 2, \cdots, N$, are passive. Let Π and Γ be a solution pair to the regulator equations (6.5). If there exists a positive definite matrix P such that

$$P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P < \mathbf{0}, \tag{6.9}$$

$$PE_f = C_f^T, \tag{6.10}$$

where $A_f = \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right]$, $B_f = \left[\begin{smallmatrix} B & \mathbf{0} \\ \mathbf{0} & I \end{smallmatrix}\right]$, $H_f = \left[\begin{smallmatrix} H & \mathbf{0} \\ \mathbf{0} & I \end{smallmatrix}\right]$, $C_f = \left[\begin{smallmatrix} C & \mathbf{0} \end{smallmatrix}\right]$, $E_f = \left[\begin{smallmatrix} E \\ \mathbf{0} \end{smallmatrix}\right]$ and $F = \left[\begin{smallmatrix} D_c & C_c \\ B_c & A_c \end{smallmatrix}\right]$, then the network of agents (6.1) with protocol (6.4a)-(6.4b), where $K = \Gamma - D_c H \Pi$, is robustly output regulated.

Proof. Let $\tilde{x}_i = x_i - \Pi w_i$, $i = 1, 2, \cdots, N$, where Π together with Γ satisfies (6.5). We get

$$\begin{cases} \dot{\tilde{x}}_1 = (A + BD_cH)\tilde{x}_1 + BC_c v_1 - E\phi_1\left(C\tilde{x}_1\right) - \Pi T(z - Rw_1) \\ \dot{v}_1 = B_c H \tilde{x}_1 + A_c v_1 \end{cases}, \tag{6.11a}$$

$$\begin{cases} \dot{\tilde{x}}_i = (A + BD_cH)\tilde{x}_i + BC_c v_i - E\phi_i\left(C\tilde{x}_i\right) - \Pi \sum_{j=1}^{N} a_{ij}(w_i - w_j) \\ \dot{v}_i = B_c H \tilde{x}_i + A_c v_i,\ i = 2, \cdots, N \end{cases}. \tag{6.11b}$$

Denote $\Sigma_1 := \Pi T(z - Rw_1)$ and $\Sigma_i := \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i)$, $i = 2, \cdots, N$. By Lemma 6.4, $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, $i = 1, 2, \cdots, N$.

Partition $P > \mathbf{0}$ in (6.9)-(6.10) appropriately as $P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$. It follows that $P_1 E = C^T$ and $P_2^T E = \mathbf{0}$. Consider a Lyapunov function candidate

$$V_o(\tilde{x}_i, v_i) = \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T P \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}, \ i = 1, 2, \cdots, N,$$

Obviously, $V_o(\tilde{x}_i, v_i)$ is positive definite and radially unbounded. Then the time derivative of $V_o(\tilde{x}_i, v_i)$ along the trajectories of (6.11a)-(6.11b) is given by

$$\begin{aligned}
&\dot{V}_o(\tilde{x}_i, v_i) \\
=&2 \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix} \left[\begin{pmatrix} A + BD_cH & BC_c \\ B_cH & A_c \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} - \begin{pmatrix} E\phi_i(C\tilde{x}_i) + \Sigma_i \\ \mathbf{0} \end{pmatrix} \right] \\
=&2 \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T P(A_f + B_f F H_f) \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} - 2\tilde{x}_i^T P_1 E \phi_i(C\tilde{x}_i) - 2v_i^T P_2^T E \phi_i(C\tilde{x}_i) \\
&- 2\tilde{x}_i^T P_1 \Sigma_i - 2v_i^T P_2^T \Sigma_i \\
=&\begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \left[P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P \right] \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} \\
&- 2(C\tilde{x}_i)^T \phi_i(C\tilde{x}_i) - 2\tilde{x}_i^T P_1 \Sigma_i - 2v_i^T P_2^T \Sigma_i \\
\leqslant&\begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \left[P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P \right] \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} \\
&- 2\tilde{x}_i^T P_1 \Sigma_i - 2v_i^T P_2^T \Sigma_i,
\end{aligned}$$

where we have used the property of passivity of each $\phi_i(\cdot)$. We note that there always exists a positive real number α such that

$$\begin{aligned}
&\dot{V}_o(\tilde{x}_i, v_i) \\
\leqslant& - \alpha \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} - 2\tilde{x}_i^T P_1 \Sigma_i - 2v_i^T P_2^T \Sigma_i \\
\leqslant& - \alpha \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} + 2\|\tilde{x}_i\| \|P_1 \Sigma_i\| + 2\|v_i\| \|P_2^T \Sigma_i\| \\
\leqslant& - \alpha\|\tilde{x}_i\|^2 - \alpha\|v_i\|^2 + \beta\|\tilde{x}_i\|^2 + \frac{1}{\beta}\|P_1\Sigma_i\|^2 + \gamma\|v_i\|^2 + \frac{1}{\gamma}\|P_2^T \Sigma_i\|^2 \\
=& - (\alpha - \beta)\|\tilde{x}_i\|^2 - (\alpha - \gamma)\|v_i\|^2 + \frac{1}{\beta}\|P_1\Sigma_i\|^2 + \frac{1}{\gamma}\|P_2^T \Sigma_i\|^2
\end{aligned}$$

$$\leqslant -\min\{\alpha-\beta, \alpha-\gamma\} \left\| \begin{matrix} \tilde{x}_i \\ v_i \end{matrix} \right\|^2 + \left(\frac{1}{\beta}\|P_1\|^2 + \frac{1}{\gamma}\|P_2^T\|^2 \right) \|\Sigma_i\|^2,$$

where $0 < \beta < \alpha$ and $0 < \gamma < \alpha$. Following a similar argument as in the proof of Theorem 6.6, this proof is completed. □

Next we will now move to the design of suitable F and $P > \mathbf{0}$ such that (6.9)-(6.10) hold.

Theorem 6.11. There exist F and $P > \mathbf{0}$ such that (6.9)-(6.10) hold if and only if there exist matrices $X > \mathbf{0}$ and $Y > \mathbf{0}$ such that $XY = I$,

$$B_f^{\perp} \left(A_f X + X A_f^T \right) B_f^{\perp T} < \mathbf{0}, \tag{6.12}$$

$$E_f = X C_f^T, \tag{6.13}$$

$$\left(H_f^T \right)^{\perp} \left(Y A_f + A_f^T Y \right) \left(H_f^T \right)^{\perp T} < \mathbf{0}. \tag{6.14}$$

In this case, a suitable P is given by $P = X^{-1}$, and a suitable F is given by

$$F = -r B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T \right)^{-1},$$

where r and Θ_x are determined by: Choose a positive real number r such that

$$\Theta_x := r B_f B_f^T - A_f X - X A_f^T > \mathbf{0}.$$

Proof. The more complex sector boundedness case will be studied in Theorem 6.16 in Subsection 6.4.2. Since the proof runs along similar lines as the proof of the analogous results for the sector boundedness case in Theorem 6.16, we will omit it here. □

Note that Theorem 6.11 is not yet entirely satisfactory in the sense that it would enable us to compute a suitable protocol. The problem is that it does not tell us how to choose the dimension n_c in the protocol state space. The following result resolves this problem.

Theorem 6.12. There exists a nonnegative integer n_c, matrices $X > \mathbf{0}$, $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ such that the conditions in Theorem 6.11 hold if and only if there exist matrices X_p, Y_p of size $n \times n$ such that $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$,

$$B^{\perp} \left(A X_p + X_p A^T \right) B^{\perp T} < \mathbf{0}, \tag{6.15}$$

$$E = X_p C^T, \tag{6.16}$$

$$\left(H^T\right)^{\perp}\left(Y_pA + A^TY_p\right)\left(H^T\right)^{\perp T} < \mathbf{0}, \tag{6.17}$$

$$Y_pE = C^T, \tag{6.18}$$

$$\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \geqslant \mathbf{0}, \tag{6.19}$$

$$\operatorname{rank}\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \leqslant n + n_c. \tag{6.20}$$

Proof. This proof is straightforward following that for Theorem 6.17. For the same reason, it is also left to the reader. □

Below we will study the case that the unknown feedback nonlinearities are assumed to be sector bounded.

6.4 Sector bounded nonlinearities

First we consider the dynamic state feedback protocol (6.3a)-(6.3b).

6.4.1 Dynamic state feedback

Theorem 6.13. Assume that $\phi_i(\cdot)$'s, $i = 1, 2, \cdots, N$, are sector bounded within $[S_1, S_2]$, where $\mathbf{0} \leqslant S_1 < S_2$. Let Π and Γ be a solution pair to (6.5). If there exists a positive definite matrix P, a matrix F and a positive real number τ such that

$$\left[\begin{array}{c:c} \begin{array}{c} P(A+BF) + (A+BF)^TP \\ -\tau C^T(S_1S_2 + S_2S_1)C \end{array} & -PE + \tau C^T(S_1+S_2) \\ \hdashline -E^TP + \tau(S_1+S_2)C & -2\tau I \end{array}\right] < \mathbf{0} \tag{6.21}$$

holds, then the network of agents (6.1) with protocol (6.3a)-(6.3b), where $K = \Gamma - F\Pi$, is robustly output regulated.

Proof. As in the proof of Theorem 6.6, choose a Lyapunov function candidate $V_2(\tilde{x}_i) = \tilde{x}_i^TP\tilde{x}_i$, $i = 1, 2, \cdots, N$, where $P > \mathbf{0}$ together with F and $\tau > 0$ satisfies (6.21). Obviously, $V_2(\tilde{x}_i)$ is positive definite and radially unbounded. Then the time derivative of $V_2(\tilde{x}_i)$ along the trajectories of (6.7a)-(6.7b) is given by

$$\dot{V}_2(\tilde{x}_i) = 2\tilde{x}_i^TP[(A+BF)\tilde{x}_i - E\phi_i(C\tilde{x}_i) - \Sigma_i]$$

$$= \begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix}^T \left[\begin{array}{c:c} P(A+BF)+ & -PE \\ (A+BF)^T P & \\ \hdashline -E^T P & \mathbf{0} \end{array}\right] \begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix} - 2\tilde{x}_i^T P \Sigma_i.$$

On the other hand, we have

$$\begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2}C^T(S_1S_2 + S_2S_1)C & -\frac{1}{2}C^T(S_1+S_2) \\ -\frac{1}{2}(S_1+S_2)C & I \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \phi_i(C\tilde{x}_i) \end{bmatrix} \leqslant \mathbf{0}$$

by using the property of sector boundedness. Thus, using (6.21), there always exists a positive real number α such that

$$\dot{V}_2(\tilde{x}_i) \leqslant -\alpha\tilde{x}_i^T P\tilde{x}_i - \alpha\phi_i(C\tilde{x}_i)^T P\phi_i(C\tilde{x}_i) - 2\tilde{x}_i^T P\Sigma_i.$$

Following a similar analysis as in the proof of Theorem 6.6, this proof can be completed. □

Similarly as in Subsection 6.3.1, we will discuss how to compute a suitable F below.

Lemma 6.14. There exists a positive definite matrix P, a matrix F and a positive real number τ such that (6.21) holds if and only if there exists a positive definite matrix Q and a positive real number ρ such that the following LMI holds:

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} \left[\begin{array}{c:c} Q\left(A - \frac{1}{2}E(S_1+S_2)C\right)^T & \\ \left(A - \frac{1}{2}E(S_1+S_2)C\right)Q & QC^T \\ +\frac{1}{2}\rho EE^T & \\ \hdashline CQ & -2\rho(S_2-S_1)^{-2} \end{array}\right] \left(\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp}\right)^T < \mathbf{0}. \quad (6.22)$$

In this case, a suitable P is given by $p = Q^{-1}$, a suitable τ is given by $\tau = \frac{1}{\rho}$, and a suitable F is given by $F = \mu B^T Q^{-1}$, where the real number μ is chosen to satisfy

$$\left[\begin{array}{c:c} Q\left(A - \frac{1}{2}E(S_1+S_2)C\right)^T & \\ \left(A - \frac{1}{2}E(S_1+S_2)C\right)Q & QC^T \\ +\frac{1}{2}\rho EE^T + 2\mu BB^T & \\ \hdashline CQ & -2\rho(S_2-S_1)^{-2} \end{array}\right] < \mathbf{0}. \quad (6.23)$$

Since the above result is slightly different from Theorem 3.9 in Chapter 3, its proof is given below.

Proof. For the ‘only if’ part, by taking the Schur complement, (6.21) is equivalent to

$$\left[\begin{array}{c:c} \begin{array}{c}(A-\frac{1}{2}E(S_1+S_2)C)^TP \\ +P(A-\frac{1}{2}E(S_1+S_2)C) \\ +F^TB^TP+PBF \\ +\frac{1}{2\tau}PEE^TP\end{array} & C^T \\ \hdashline C & -\frac{2}{\tau}(S_2-S_1)^{-2}\end{array}\right] < \mathbf{0}. \tag{6.24}$$

Let $Q = P^{-1}$ and $\rho = \frac{1}{\tau}$. Then we get

$$\left[\begin{array}{c:c} \begin{array}{c}Q(A-\frac{1}{2}E(S_1+S_2)C)^T \\ +(A-\frac{1}{2}E(S_1+S_2)C)Q \\ +QF^TB^T+BFQ \\ +\frac{1}{2}\rho EE^T\end{array} & QC^T \\ \hdashline CQ & -2\rho(S_2-S_1)^{-2}\end{array}\right] < \mathbf{0}.$$

It is easily verified that

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} B^{\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}. \tag{6.25}$$

By premultiplying with (6.25) and postmultiplying with the transpose of (6.25), (6.22) is obtained.

For the 'if' part, again by taking the Schur complement, (6.22) implies

$$B^{\perp}\left[Q\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T + \left(A-\frac{1}{2}E(S_1+S_2)\right.\right.$$
$$\left.\left. C\right)Q + \frac{1}{2}\rho EE^T + \frac{1}{2\rho}QC^T(S_2-S_1)^2CQ\right]\left(B^{\perp}\right)^T < \mathbf{0}.$$

By Finsler's lemma [24], it follows that there exists a real number μ such that

$$Q\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T + \left(A-\frac{1}{2}E(S_1+S_2)C\right)Q$$
$$+\frac{1}{2}\rho EE^T + \frac{1}{2\rho}QC^T(S_1-S_2)^2CQ + 2\mu BB^T < \mathbf{0},$$

i.e. (6.23). Let $P = Q^{-1}$, $\tau = \frac{1}{\rho}$ and $F := \mu B^T P$. Then we get

$$\left(A-\frac{1}{2}E(S_1+S_2)C\right)^T P + P\left(A-\frac{1}{2}E(S_1+S_2)C\right)$$
$$+F^TB^TP + PBF + \frac{1}{2\tau}PEE^TP + \frac{1}{2}\tau C^T(S_2-S_1)^2C < \mathbf{0},$$

i.e. (6.24) holds. This completes the proof. □

Thus, the computation of a cooperative robust output regulation protocol can be performed as follows:

- Compute a solution pair (Π, Γ) to (6.5);
- Compute a $Q > \mathbf{0}$ and a $\rho > 0$ such that (6.22) holds;
- Compute a μ such that (6.23) holds;
- Compute $F := \mu B^T Q^{-1}$;
- Compute $K := \Gamma - F\Pi$.

6.4.2 Dynamic output feedback

In this subsection, we will discuss the design of a dynamic output feedback protocol in the form of (6.4a)-(6.4b).

Theorem 6.15. Assume that $\phi_i(\cdot)$'s, $\forall\, i = 1, 2, \cdots, N$, are sector bounded. Let Π and Γ be a solution pair to the regulator equations (6.5). If there exists a positive definite matrix P and a positive real number τ such that

$$\left[\begin{array}{c:c} \begin{array}{c} P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P \\ -\tau C_f^T (S_1 S_2 + S_2 S_1) C_f \end{array} & -PE_f + \tau C_f^T (S_1 + S_2) \\ \hdashline -E_f^T P + \tau (S_1 + S_2) C_f & -2\tau I \end{array}\right] < \mathbf{0}, \quad (6.26)$$

where $A_f = \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right]$, $B_f = \left[\begin{smallmatrix} B & \mathbf{0} \\ \mathbf{0} & I \end{smallmatrix}\right]$, $H_f = \left[\begin{smallmatrix} H & \mathbf{0} \\ \mathbf{0} & I \end{smallmatrix}\right]$, $C_f = \left[\begin{smallmatrix} C & \mathbf{0} \end{smallmatrix}\right]$, $E_f = \left[\begin{smallmatrix} E \\ \mathbf{0} \end{smallmatrix}\right]$ and $F = \left[\begin{smallmatrix} D_c & C_c \\ B_c & A_c \end{smallmatrix}\right]$, then the network of agents (6.1) with protocol (6.4a)-(6.4b), where $K = \Gamma - D_c H \Pi$, is robustly output regulated.

Proof. As in the proof of Theorem 6.10, partition $P > \mathbf{0}$ in (6.26) appropriately as $P = \left[\begin{smallmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{smallmatrix}\right]$. It follows that $P_1 E = C^T$ and $P_2^T E = \mathbf{0}$ as well. Consider a Lyapunov function candidate

$$V_t\left(\tilde{x}_i, v_i\right) = \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T P \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}, \; i = 1, 2, \cdots, N,$$

Obviously, $V_t\left(\tilde{x}_i, v_i\right)$ is positive definite and radially unbounded. Then the time derivative of $V_t\left(\tilde{x}_i, v_i\right)$ along the trajectories of (6.11a)-(6.11b) is given by

$$\begin{aligned} &\dot{V}_t\left(\tilde{x}_i, v_i\right) \\ = &\begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix}^T \left[P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P\right] \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&- 2\begin{pmatrix}\tilde{x}_i\\ v_i\end{pmatrix}^T \begin{bmatrix}P_1E\\ P_2^TE\end{bmatrix}\phi_i\left(C\tilde{x}_i\right) - 2\tilde{x}_i^T P_1\Sigma_i - 2v_i^T P_2^T\Sigma_i\\
=&\begin{pmatrix}\tilde{x}_i\\ v_i\\ \phi_i\left(C\tilde{x}_i\right)\end{pmatrix}^T \begin{bmatrix}P(A_f+B_fFH_f)+(A_f+B_fFH_f)^TP & -PE_f\\ -E_f^TP & \mathbf{0}\end{bmatrix}\begin{pmatrix}\tilde{x}_i\\ v_i\\ \phi_i\left(C\tilde{x}_i\right)\end{pmatrix}\\
&- 2\tilde{x}_i^T P_1\Sigma_i - 2v_i^T P_2^T\Sigma_i.
\end{aligned}
$$

On the other hand, we have

$$
\begin{bmatrix}\tilde{x}_i\\ v_i\\ \phi_i(C\tilde{x}_i)\end{bmatrix}^T \begin{bmatrix}\frac{1}{2}C_f^T(S_1S_2+S_2S_1)C_f & -\frac{1}{2}C_f^T(S_1+S_2)\\ -\frac{1}{2}(S_1+S_2)C_f & I\end{bmatrix}\begin{bmatrix}\tilde{x}_i\\ v_i\\ \phi_i(C\tilde{x}_i)\end{bmatrix} \leqslant \mathbf{0}
$$

by using the property of sector boundedness. Thus, using (6.26), there always exists a positive real number α such that

$$
\begin{aligned}
\dot{V}_2(\tilde{x}_i) \leqslant &- \alpha\tilde{x}_i^T\tilde{x}_i - \alpha v_i^Tv_i - \alpha\phi_i(C\tilde{x}_i)^T\phi_i(C\tilde{x}_i)\\
&- 2\tilde{x}_i^TP_1\Sigma_i - 2v_i^TP_2^T\Sigma_i.
\end{aligned}
$$

Following a similar analysis as in the proof of Theorem 6.10, this proof can be completed. □

Next we will now move to the design of suitable F, $P > \mathbf{0}$ and $\tau > 0$ such that (6.26) holds.

Theorem 6.16. There exist F, $P > \mathbf{0}$ and $\tau > 0$ such that (6.26) holds if and only if there exist matrices $X > \mathbf{0}$, $Y > \mathbf{0}$ and positive real numbers α, β such that $XY = I$, $\alpha\beta = 1$,

$$
\begin{bmatrix}B_f\\ \mathbf{0}\end{bmatrix}^{\perp}\left[\begin{array}{c:c}\begin{array}{c}\left(A_f-\frac{1}{2}E_f(S_1+S_2)C_f\right)X\\ +X\left(A_f-\frac{1}{2}E_f(S_1+S_2)C_f\right)^T\\ +\frac{1}{2}\alpha E_fE_f^T\end{array} & XC_f^T\\ \hdashline C_fX & -2\alpha(S_2-S_1)^{-2}\end{array}\right]\begin{bmatrix}B_f\\ \mathbf{0}\end{bmatrix}^{\perp T} < \mathbf{0}, \tag{6.27}
$$

$$
\begin{bmatrix}H_f^T\\ \mathbf{0}\end{bmatrix}^{\perp}\left[\begin{array}{c:c}\begin{array}{c}Y\left(A_f-\frac{1}{2}E_f(S_1+S_2)C_f\right)\\ +\left(A_f-\frac{1}{2}E_f(S_1+S_2)C_f\right)^TY\\ +\frac{1}{2}\beta C_f^T(S_2-S_1)^2C_f\end{array} & YE_f\\ \hdashline E_f^TY & -2\beta I_p\end{array}\right]\begin{bmatrix}H_f^T\\ \mathbf{0}\end{bmatrix}^{\perp T} < \mathbf{0}. \tag{6.28}
$$

In this case, a suitable P is given by $P = X^{-1}$, a suitable τ is given by $\tau = \alpha^{-1}$, and a suitable F is given by

$$F = -rB_f^T\Theta_x^{-1}XH_f^T\left(H_fX\Theta_x^{-1}XH_f^T\right)^{-1},$$

where r and Θ_x are determined by: Choose a positive real number r such that

$$\Theta_x := rB_fB_f^T - Q_X - \frac{1}{2}\alpha E_fE_f^T - \frac{1}{2\alpha}XC_f^T(S_2-S_1)^2C_fX > \mathbf{0}, \tag{6.29}$$

where $Q_X := \left[A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right]X + X\left[A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right]^T$.

Proof. The existence of solutions F, $P > \mathbf{0}$ and $\tau > 0$ to (6.26) is equivalent to the existence of F, $X > \mathbf{0}$ and $\alpha > 0$ such that

$$\begin{bmatrix} B_fFH_fX + (B_fFH_fX)^T + Q_X + \frac{1}{2}\alpha E_fE_f^T & XC_f^T \\ C_fX & -2\alpha(S_2-S_1)^{-2} \end{bmatrix} < \mathbf{0}. \tag{6.30}$$

This can be seen by taking $X = P^{-1}$, $\alpha = \tau^{-1}$, and considering the appropriate Schur complements.

(only if) Let X be a positive definite solution to (6.30). Define $Y = X^{-1}$. Then $X > \mathbf{0}$, $Y > \mathbf{0}$ and $XY = I$. Obviously, (6.27) holds. (6.30) implies that

$$\begin{bmatrix} YB_fFH_f + (YB_fFH_f)^T + Q_Y + \frac{1}{2}\beta C_f^T(S_2-S_1)^2C_f & YE_f \\ E_f^TY & -2\beta I \end{bmatrix} < \mathbf{0},$$

where $Q_Y := Y\left[A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right] + \left[A_f - \frac{1}{2}E_f(S_1+S_2)C_f\right]^TY$, $\beta = \alpha^{-1}$, which then implies (6.28).

(if) By Finsler's lemma [24], (6.27) implies that there exists a $r > 0$ such that

$$\begin{bmatrix} rB_fB_f^T - Q_X - \frac{1}{2}\alpha E_fE_f^T & -XC_f^T \\ -C_fX & 2\alpha(S_2-S_1)^{-2} \end{bmatrix} > \mathbf{0},$$

equivalently, $\Theta_x > \mathbf{0}$. Similarly, (6.28) implies that there exists a matrix $S > \mathbf{0}$ such that

$$\begin{bmatrix} H_f^TS^{-1}H_f - Q_Y - \frac{1}{2}\beta C_f^T(S_2-S_1)^2C_f & -YE_f \\ -E_f^TY & 2\beta I \end{bmatrix} > \mathbf{0},$$

equivalently, $\Theta_y := XH_f^TS^{-1}H_fX - Q_X - \frac{1}{2\alpha}XC_f^T(S_2-S_1)^2C_fX - \frac{1}{2}\alpha E_fE_f^T > \mathbf{0}$.

Define

$$\Xi := rI - r^2B_f^T\Theta_x^{-1}B_f + r^2B_f^T\Theta_x^{-1}XH_f^T\left(H_fX\Theta_x^{-1}XH_f^T\right)^{-1}H_fX\Theta_x^{-1}B_f,$$

where $H_fX\Theta_x^{-1}XH_f^T$ is positive definite since $\Theta_x > \mathbf{0}$, $X > \mathbf{0}$ and H_f has full row rank. Obviously, $\Xi > \mathbf{0}$ if and only if there exits a matrix $Z > \mathbf{0}$ such that

$$rI - r^2 B_f^T \Theta_x^{-1} B_f + r^2 B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T + Z\right)^{-1} H_f X \Theta_x^{-1} B_f > \mathbf{0},$$

or equivalently, using the matrix inversion lemma [56],

$$rI - r^2 B_f^T \left(\Theta_x + X H_f^T Z^{-1} H_f X\right)^{-1} B_f > \mathbf{0},$$

equivalently, using the Schur complement lemma,

$$\begin{bmatrix} rI & rB_f^T \\ rB_f & \Theta_x + X H_f^T Z^{-1} H_f X \end{bmatrix} > \mathbf{0},$$

and equivalently,

$$\begin{aligned} &\Theta_x + X H_f^T Z^{-1} H_f X - r B_f B_f^T \\ =& - Q_X - \frac{1}{2}\alpha E_f E_f^T - \frac{1}{2\alpha} X C_f^T (S_2 - S_1)^2 C_f X + X H_f^T Z^{-1} H_f X > \mathbf{0}. \end{aligned}$$

Above it has been shown that this holds if we take $Z = S$, in this case the above inequality is exactly $\Theta_y > \mathbf{0}$. This shows that $\Xi > \mathbf{0}$.

Now, clearly,

$$\begin{aligned} &\left[F + r B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1}\right] H_f X \Theta_x^{-1} X H_f^T \\ &\left[F + r B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1}\right]^T = \mathbf{0} < \Xi \end{aligned}$$

for the particular choice $F = -r B_f^T \Theta_x^{-1} X H_f^T \left(H_f X \Theta_x^{-1} X H_f^T\right)^{-1}$. The latter inequality holds if and only if

$$\left(r B_f^T + F H_f X\right) \Theta_x^{-1} \left(r B_f^T + F H_f X\right)^T < rI,$$

which in turn is equivalent to

$$\begin{bmatrix} \Theta_x & \left(r B_f^T + F H_f X\right)^T \\ r B_f^T + F H_f X & rI \end{bmatrix} > \mathbf{0},$$

and to

$$\begin{aligned} &\frac{1}{r}\left(r B_f^T + F H_f X\right)^T \left(r B_f^T + F H_f X\right) < \Theta_x \\ =& r B_f B_f^T - Q_X - \frac{1}{2}\alpha E_f E_f^T - \frac{1}{2\alpha} X C_f^T (S_2 - S_1)^2 C_f X. \end{aligned}$$

It follows that

$$B_f F H_f X + (B_f F H_f X)^T + \frac{1}{r} X H_f^T F^T F H_f X$$
$$+ Q_x + \frac{1}{2}\alpha E_f E_f^T + \frac{1}{2\alpha} X C_f^T (S_2 - S_1)^2 C_f X < \mathbf{0},$$

which yields (6.30) since $\frac{1}{r} X H_f^T F^T F H_f X$ is always positive semi-definite. This completes the proof. □

Note that Theorem 6.16 is not yet entirely satisfactory in the sense that it would enable us to compute a suitable protocol. The problem is that it does not tell us how to choose the dimension n_c in the protocol state space. The following result resolves this problem.

Theorem 6.17. There exists a nonnegative integer n_c, matrices $X > \mathbf{0}$, $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ and positive real numbers $\alpha > 0$, $\beta > 0$ such that the conditions in Theorem 6.16 hold if and only if there exist matrices X_p, Y_p of size $n \times n$ and $\alpha > 0$, $\beta > 0$ such that $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$, $\alpha\beta = 1$,

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp} \begin{bmatrix} Q_x + \frac{1}{2}\alpha E E^T & X_p C^T \\ C X_p & -2\alpha (S_2 - S_1)^{-2} \end{bmatrix} \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \tag{6.31}$$

$$\begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix}^{\perp} \begin{bmatrix} Q_y + \frac{1}{2}\beta C^T (S_2 - S_1)^2 C & Y_p E \\ E^T Y_p & -2\beta I \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \tag{6.32}$$

$$\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \geqslant \mathbf{0}, \tag{6.33}$$

$$\operatorname{rank} \begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \leqslant n + n_c, \tag{6.34}$$

where

$$Q_x := \left[A - \frac{1}{2}E(S_1 + S_2)C\right] X_p + X_p \left[A - \frac{1}{2}E(S_1 + S_2)C\right]^T,$$

$$Q_y := Y_p \left[A - \frac{1}{2}E(S_1 + S_2)C\right] + \left[A - \frac{1}{2}E(S_1 + S_2)C\right]^T Y_p.$$

Proof. (only if) Assume that there exists a nonnegative integer n_c, $X > \mathbf{0}$, $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ and $\alpha > 0$, $\beta > 0$ such that $XY = I$, $\alpha\beta = 1$, (6.27) and (6.28) hold. Partition

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}, \quad Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}.$$

Note that $B_f^{\perp} = \begin{bmatrix} B^{\perp} & \mathbf{0} \end{bmatrix}$, $H_f^{T\perp} = \begin{bmatrix} H^{T\perp} & \mathbf{0} \end{bmatrix}$, $\begin{bmatrix} B_f \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} B_f^{\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$, $\begin{bmatrix} H_f^T \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} H_f^{T\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$. In this way we obtain (6.31) and (6.32). $XY = I$ implies that $X_pY_p + X_{pc}Y_{pc}^T = I$ and $X_pY_{pc} + X_{pc}Y_c = \mathbf{0}$. Thus

$$Y_p - X_p^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T \geqslant \mathbf{0}. \tag{6.35}$$

Using the Schur complement, this is equivalent to (6.33). In addition,

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} &= \operatorname{rank}(X_p) + \operatorname{rank}\left(Y_p - X_p^{-1}\right) \\ &= n + \operatorname{rank}\left(Y_{pc}Y_c^{-1}Y_{pc}^T\right) \leqslant n + n_c. \end{aligned}$$

So (6.34) holds.

(if) Let Y_{pc} and $Y_c > \mathbf{0}$ by any matrices satisfying (6.35) while $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$, $\alpha > 0$ and $\beta > 0$ satisfy (6.31), (6.32), (6.33) and $\alpha\beta = 1$, respectively, and n_c is chosen so that (6.34) is satisfied. It can be verified that a matrix pair (X, Y) such that

$$Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}, \quad X = Y^{-1}$$

together with the above α and β satisfy the conditions in Theorem 6.16. This completes the proof. □

Thus, the computation of a cooperative robust regulation protocol can be performed as follows:

- Compute a solution pair (Π, Γ) to (6.5);
- Compute $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$, $\alpha > 0$ and $\beta > 0$ through (6.31), (6.32), (6.33) and $\alpha\beta = 1$;
- Choose n_c as $n_c = \operatorname{rank}\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} - n$, then define A_f, B_f, C_f, E_f and H_f as introduced before in the statement of Theorem 6.14;
- Choose $Y_c > \mathbf{0}$ and Y_{pc} satisfying (6.35), consequently, $Y > \mathbf{0}$ and $X > \mathbf{0}$ are obtained;
- Compute $r > 0$ such that $\Theta_x > \mathbf{0}$;
- Definte F by $F = -rB_f^T\Theta_x^{-1}XH_f^T\left(H_fX\Theta_x^{-1}XH_f^T\right)^{-1}$;
- Partition F as $\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$;
- Compute $K = \Gamma - D_cH\Pi$.

6.5 Discussions

In this section we will elaborate on some of the conditions that were obtained in the previous two sections.

1. Note that in (6.5) the condition $\mathbf{0} = C\Pi$ is imposed, which is in fact a necessary requirement in our robust output regulation problem. This can be shown by contradiction. Suppose that $C\Pi \neq \mathbf{0}$ and take the closed-loop dynamics of agent 1 as the example. By also defining $\tilde{x}_1 = x_1 - \Pi w_1$, where Π satisfies (6.5), we get

$$\dot{\tilde{x}}_1 = (A + BF)\tilde{x}_1 - E\phi_1\left(C\tilde{x}_1 + C\Pi w_1\right) - \Pi T(z - Rw_1),$$

which implies that $E\phi_1(C\Pi w_1) = \mathbf{0}$ as long as w_1 and $\tilde{x}_1$ reach w and zero, respectively. However, since $\phi_1(\cdot)$ is passive or sector bounded and w_1 approaching w is a persistently exciting signal, $E\phi_1(C\Pi w_1) = \mathbf{0}$ cannot always hold. We have the same argument for the other agents. In other words, the condition $\mathbf{0} = C\Pi$ guarantees that the unknown nonlinearities $\phi_i(\cdot)$, $\forall\ i = 1, 2, \cdots, N$, vanish when the cooperative output regulation is achieved. Otherwise, we have to solve nonlinear regulator equations involving the unknown nonlinearities, which is impossible. The necessity as well as an example regarding output regulation of a specific Lur'e system has been discussed in [43].

2. The feasibility of the regulator equations (6.5) is critical in our protocol design. By applying Theorem 9.8 in [64], necessary and sufficient conditions for solvability of the regulator equations (6.5) are obtained below. For the sake of completeness, Theorem 9.8 in [64] is recalled:

Lemma 6.18. Consider the linear matrix equation

$$\sum_{i=1}^{k} A_i X q_i(B) = C, \tag{6.36}$$

where A_i, B and C are given matrices with B square, $q_i(s)$'s are real polynomials and X is unknown. (6.36) has a solution X if and only if the polynomial matrices

$$\begin{bmatrix} A(s) & \mathbf{0} \\ \mathbf{0} & sI - B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(s) & C \\ \mathbf{0} & sI - B \end{bmatrix}$$

have the same Smith form. Here, the polynomial matrix $A(s)$ is defined by $A(s) = \sum_{i=1}^{k} q_i(s) A_i$.

By applying Lemma 6.18 with $k = 2$, $B = S$, $q_1(s) = 1$, $q_2(s) = s$,

$$A_1 = \begin{bmatrix} -A & -B \\ C & \mathbf{0} \\ H & \mathbf{0} \end{bmatrix}, A_2 = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, X = \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix}, C = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ R \end{bmatrix},$$

and hence

$$A(s) = \begin{bmatrix} sI - A & -B \\ C & \mathbf{0} \\ H & \mathbf{0} \end{bmatrix},$$

we obtain the following result:

Corollary 6.19. The regulator equations (6.5) have a solution pair (Π, Γ) if and only if

$$\begin{bmatrix} sI - A & -B & \mathbf{0} \\ C & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & sI - S \end{bmatrix} \text{ and } \begin{bmatrix} sI - A & -B & \mathbf{0} \\ C & \mathbf{0} & \mathbf{0} \\ H & \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} & sI - S \end{bmatrix}$$

have the same Smith form.

3. The feasibility of (6.8) and (6.22) has been discussed in [74]. In general, it can only be checked numerically whether these have solutions.

6.6 Simulation Examples

In this section we present some numerical simulations to illustrate the results obtained in this chapter. Due to space limitations, we only consider the sector boundedness case in which the dynamics of agents is described by the following nonlinear ordinary differential equations:

$$\begin{cases} \dot{x} = Ax + Bu + Ed \\ y = Cx \\ z = Hx \\ d = -\phi(y) \end{cases}, \tag{6.37}$$

where $A = [-3.2, 10, 0; 1, -1, 1; 0, -14.87, 0]$, $B = [1, 0; 1, 0; 0, 1]$, $C = [1, 0, 0]$, $E = [-2.95; 0; 0]$ and $H = [0, 0, 1]$. The nonlinearity $\phi_i(\cdot)$ for agent i is taken as $(0.4i - 0.1) \cdot \text{atan}(\cdot)$, $i = 1, 2, 3, 4$, where 'atan' denotes the arctangent function. It is easily checked that $i \cdot \text{atan}(\cdot)$, $\forall\, i = 1, 2, 3, 4$, is sector bounded with $S_1 = 0$ and $S_2 = 2$, see Fig. 6.2. Consider a network of 4 such agents as shown in Fig. 6.3, where agent 1 has access to the exosystem. The dotted edge $(3, 1)$ means that agent

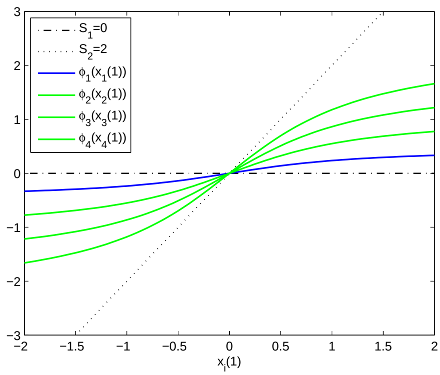

Figure 6.2: *The nonlinearities*

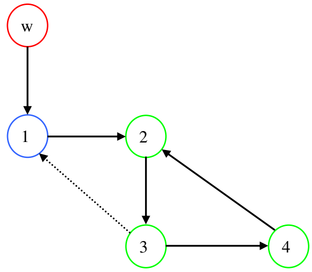

Figure 6.3: *The interconnection topology*

1 can get the relative information with respect to agent 3 but will not use it in its protocol. The exosystem is taken as

$$\dot{w} = Sw, \quad z = Rw, \tag{6.38}$$

where $S = [0, 1; -1, 0]$ and $R = [1, 0]$. A suitable T can be chosen as $[2; 1]$ such that $S - TR$ is Hurwitz. Choose the initial conditions for w and w_i as $[1; 1]$ and $[i+1; i+1]$, $i = 1, 2, 3, 4$, respectively. Using Matlab, the trajectories of the estimates w_i's are plotted in Fig. 6.4. Clearly, all the estimates converge to the state w of the exosystem.

Example 1. We first consider the dynamic state feedback case. Now we just need to compute the feedback and feedforward gain matrices. Using the LMI

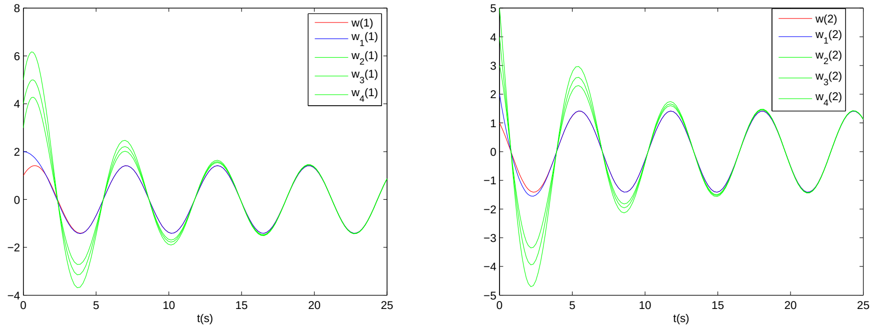

Figure 6.4: *The plots of* $w_i(t)$

Control Toolbox in Matlab, we can easily find a solution to (6.5):

$$\Pi = \begin{bmatrix} 0 & 0 \\ 0.0902 & -0.0082 \\ 1 & 0 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} -0.9016 & 0.082 \\ 1.3407 & 0.8781 \end{bmatrix},$$

and obtain the design parameters in Lemma 6.14:

$$Q = \begin{bmatrix} 35.9811 & 19.0388 & -0.8627 \\ 19.0388 & 36.1012 & 0.8627 \\ -0.8627 & 0.8627 & 55.08 \end{bmatrix},$$

$\rho = 69.0989$, $\mu = -1497.4$,

$$F = \begin{bmatrix} -27.2814 & -27.089 & -0.003 \\ -1.383 & 1.38 & -27.2283 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.5438 & -0.1401 \\ 28.4447 & 0.8894 \end{bmatrix}.$$

Choosing the initial conditions for agent i as $(i+1)\cdot[1;1;1]$, $i = 1, 2, 3, 4$, their trajectories are plotted in Fig. 6.5. All the outputs can track the reference signal generated by the exosystem in presence of heterogeneous unknown nonlinearities.

Example 2. In this example, the dynamic output feedback case will be considered. By implementing the computation procedure presented after Theorem 6.17, all the design parameters can be obtained successively. First, $\alpha > 0$ and $X_p > \mathbf{0}$

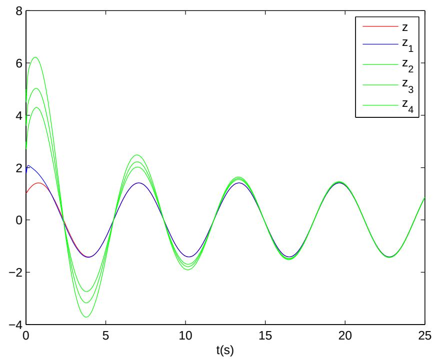

Figure 6.5: *The plots of $z_i(t)$ for Example 1*

can be computed to be $\alpha = 69.0989$ and, respectively,

$$X_p = \begin{bmatrix} 35.9811 & 19.0388 & -0.8627 \\ 19.0388 & 36.1012 & 0.8627 \\ -0.8627 & 0.8627 & 55.0800 \end{bmatrix}.$$

So, $\beta = 0.0145$ and $Y_p > \mathbf{0}$ is computed to be

$$Y_p = \begin{bmatrix} 0.0788 & -1.0095 & 1.2935 \\ -1.0095 & 41.6257 & 12.1358 \\ 1.2935 & 12.1358 & 291.9815 \end{bmatrix}.$$

Then the internal state dimension n_c in a possible protocol can be $n_c = 3$. Without loss of generality, we can choose $Y_c > \mathbf{0}$ as $Y_c = I_3$. It follows that

$$Y_{pc} = \begin{bmatrix} 0.2006 & 0 & 0 \\ -4.9316 & 4.1553 & 0 \\ 6.4449 & 10.5695 & 11.7776 \end{bmatrix}$$

from (6.35) by using Cholesky decomposition. Thus, a suitable $Y > \mathbf{0}$ is

$$Y = \begin{bmatrix} 0.0788 & -1.0095 & 1.2935 & 0.2006 & 0 & 0 \\ -1.0095 & 41.6257 & 12.1358 & -4.9316 & 4.1553 & 0 \\ 1.2935 & 12.1358 & 291.9815 & 6.4449 & 10.5695 & 11.7776 \\ 0.2006 & -4.9316 & 6.4449 & 1.0000 & 0 & 0 \\ 0 & 4.1553 & 10.5695 & 0 & 1.0000 & 0 \\ 0 & 0 & 11.7776 & 0 & 0 & 1.0000 \end{bmatrix},$$

and consequently, a suitable $X > \mathbf{0}$ is

$$X = 10^3 \begin{bmatrix} 0.0360 & 0.0190 & -0.0009 & 0.0922 & -0.0700 & 0.0102 \\ 0.0190 & 0.0361 & 0.0009 & 0.1687 & -0.1591 & -0.0102 \\ -0.0009 & 0.0009 & 0.0551 & -0.3506 & -0.5858 & -0.6487 \\ 0.0922 & 0.1687 & -0.3506 & 3.0736 & 3.0044 & 4.1287 \\ -0.0700 & -0.1591 & -0.5858 & 3.0044 & 6.8534 & 6.8988 \\ 0.0102 & -0.0102 & -0.6487 & 4.1287 & 6.8988 & 7.6412 \end{bmatrix}.$$

Now $r > 0$ can be computed as $r = 158480$ and thus $\Theta_x > \mathbf{0}$ is also known. By computing F and partitioning it appropriately, we obtain, fininally,

$$A_c = 10^5 \begin{bmatrix} 0.0189 & 0.0234 & 1.0265 \\ -0.0123 & -0.0151 & 1.1271 \\ 0.0000 & 0.0000 & -1.5848 \end{bmatrix},$$

$$B_c = 10^6 \begin{bmatrix} 1.2461 \\ 1.3038 \\ -1.8665 \end{bmatrix},$$

$$C_c = 10^4 \begin{bmatrix} 0.0368 & 0.0454 & 0.4685 \\ -0.0029 & -0.0040 & -1.2546 \end{bmatrix},$$

$$D_c = 10^5 \begin{bmatrix} 0.6236 \\ -1.4840 \end{bmatrix}.$$

Therefore, $K = \Gamma - D_c H \Pi$ is computed as

$$K = 10^5 \begin{bmatrix} -0.6236 & 0.0000 \\ 1.4840 & 0.0000 \end{bmatrix}.$$

By using the same initial conditions as in Example 1 and let the initial state for v_i, $i = 1, 2, 3, 4$, to be zero, we get the trajectories that are plotted in Fig. 6.6. Here we use two different time scales in order to make the synchronization behavior be clear at the beginning as well. Although our designed dynamic output feedback protocol can also regulate the heterogeneous Lur'e network as the above dynamic state feedback protocol, its synchronization progress is a little bit different.

6.7 Conclusions

In this chapter we have studied the cooperative robust output regulation problem for directed networks of identical Lur'e systems. The networks are allowed to be heterogeneous in the sense that the Lur'e-type nonlinearities are allowed to

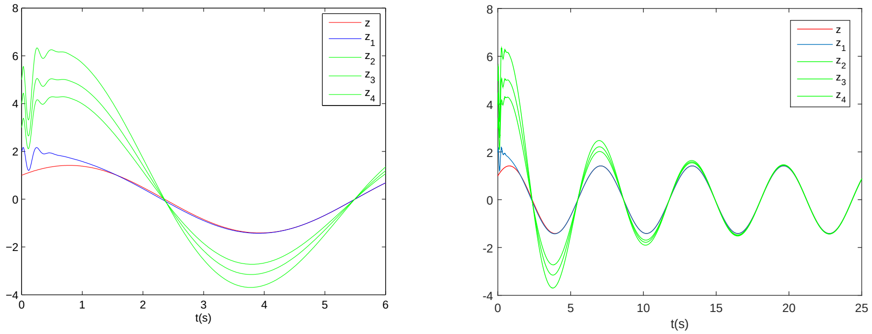

Figure 6.6: *The plots of $z_i(t)$ in Example 2*

differ for distinct agents. By designing a fully distributed estimator, a copy of the reference signal generated by the exosystem is made asymptotically at each agent and thus these agents can asymptotically track the reference signal locally. Furthermore, the protocol for each agent can also be designed locally via a suitable Lyapunov argument. Both the dynamic state feedback and the dynamic output feedback cases have been considered. In the near future we will study the case that the linear part of a Lur'e system is uncertain as well as its nonlinearity.